Applications of FORMAC in the Mathematics of General Relativity

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The use of FORMAC (formula manipulation by computer) to perform symbolic analysis on the computer in the field of general relativity is described. FORMAC will accept symbolic analytical expressions involving complex algebraic functions, elementary transcendental functions, user-defined functions, and unspecified functions of any number of dependent variables. In particular, FORMAC programs have now been written to implement the noncommutative calculus of exterior differential forms. Applications of this new capability to Hamiltonian cosmology and the theory of partial differential equations are discussed.

Introduction

In recent months the IBM FORMAC (formula manipulation by computer) system has become available at JPL for use with the IBM 360-75 computer. The present version of FORMAC is imbedded in the PL/1 programming language and represents a considerable improvement over the original FORTRAN version that was developed in 1965. FORMAC will accept symbolic analytical expressions involving complex algebraic functions, elementary transcendental functions, user-defined functions, and unspecified functions of any number of dependent variables. Instructions are available to manipulate these expressions algebraically, make substitutions and evaluations, take partial derivatives of any order, analyze the structure of generated expressions, print out results in an edited form (with raised exponents, for example) which is quite close to standard algebraic format, and punch intermediate expressions on cards in an appropriate form for later input.

The system performs a great deal of automatic simplification of expressions as they are generated. All transformations requiring the factoring of an expression, however, are under control of the user and *not* done automatically. As a result many elementary simplifications are missed if

not specifically programmed. In a complicated program this can lead to an explosive growth in the length of expressions, rapidly filling all available core. This is the major difficulty in using FORMAC and can be a serious drawback. However, our experience has been that it can usually be handled by careful and sometimes incremental programming.

Applications in General Relativity

The mathematics of general relativity is an excellent example of a field in which a large amount of tedious, time-consuming, and error-prone analysis must be performed. The prototypical problem is the derivation of the Christoffel symbols and Riemann tensor components from a given metric tensor. A number of FORMAC programs to accomplish this task have been written by users in the past few years; it is a straightforward and very satisfactory application of FORMAC.

The search for exact solutions of the field equations of general relativity, however, often requires less straightforward, but still very lengthy, analysis and several specialized FORMAC programs to aid in accomplishing this work have now been developed. Most recent investigations in this area involve the use of modern coordinate-independent formulations of general relativity such as exterior differential forms and the closely related tetrad formalisms, one of the latter being the dyadic formalism originated at JPL.

Differential Forms and the EXTCALC Subroutines

Actually, the language of exterior differential forms has found wide application in various fields of mathematics and theoretical physics other than general relativity. Examples are: Hamiltonian mechanics, differential geometry, and the general theory of sets of partial differential equations (see Reference 1). FORMAC is not immediately competent to deal with this language, which involves noncommutative operations. Recently, however, a set of PL/1 FORMAC subroutines (the EXTCALC package) was developed by F. Ernst of the Illinois Institute of Technology to implement this capability for calculations in general relativity. In collaboration with Professor Ernst, we have now extended the original EXTCALC package to incorporate additional operations so that it is no longer limited to a specific application, but is capable of performing virtually all the analytical operations of the exterior calculus of differential forms. Thus, quite general problems can now be formulated in this language and analyzed on the computer.

A thorough description and user's manual for the original EXTCALC package can be found in Reference 2, and only a brief summary of the

extended package is presented here. The specific operations of the exterior calculus implemented within the present EXTCALC package are:

Operation	Subroutine name
Exterior product	HAT
Vector contraction	CONTRACT
Exterior derivative	D or (@)
Lie derivative	LIE

One writes any symbolic differential form expression containing these operations (which may be nested) as a character string argument of the master subroutine #LET, in direct analogy with standard FORMAC statements. This master routine then directs the processing of the expression, calling the necessary operational subroutines, until a correct expression, interpretable by standard FORMAC, is created and returned. Several other convenient (though unessential) operations can also be performed within #LET. Thus, for example, one can order differential form expressions into a specified format, extract the coefficient of any basis form, and switch to a new set of basis forms.

Several diverse applications of this capability have been made in recent months. Within general relativity we have used the EXTCALC package to aid in developing a unified Hamiltonian theory of a class of homogeneous, anisotropic cosmological models—those with symmetry groups of Bianchi types VIII and IX. The differential geometry of such spaces is best treated in differential form language with anholonomic basis forms adapted to the appropriate symmetry group, which is an ideal application of the formalism.

Another application has been to perform some of the calculations involved in deriving symmetry groups of partial differential systems. Here again, as shown in the work reported in Reference 1, the use of differential forms permits the most elegant and concise formulation of the general theory of symmetry properties of sets of partial differential equations. In this approach the equations for the generators of the symmetry group are expressed in terms of the Lie derivatives of a set of exterior differential forms equivalent to the set of partial differential equations. To work out the content of these equations in a complex case can require a considerable amount of analysis. Using the FORMAC–EXTCALC capability for Lie differentiation of forms, it is now a simple matter to design a program (ISOVECT) to accomplish the analysis on the computer.

References

1. Harrison, B. K., and Estabrook, F. B., "Geometric Approach to Invariance Groups and Solution of Partial Differential Systems," *J. Math. Phys.*, Vol. 12, Feb. 1971 (in press).

2. Ernst, F. J., "Manipulation of Differential Forms on a Digital Computer," in *Proceedings of the Relativity Seminar*, Illinois Institute of Technology, Chicago, Ill., Sept. 1969.